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2021年11月15日　午前 8:05

curl $(\vec{v}) = \nabla \times \vec{v}$　　div $(\vec{v}) = \nabla \cdot \vec{v}$

$\vec{v} = (P, Q, R)$

prop: ① curl $(\nabla f) = 0$ and ② div (curl $(\vec{v})$) $= 0$.

Note ① the divergence of a vector field calculates "how badly does the v.f want to leave a bounded set."

② the curl itself is a measure of "how swirly" a v.f wants to be...

→ the curl itself is "swirly" thing.

Recasting Green's theorem.

Let $\vec{v} = \langle P, Q, 0 \rangle$ have cts partial derivatives on an open region R containing D, when D is a closed region w/ a piecewise-smooth boundary curve.

Then $\underline{\iint_D \text{curl}(\vec{v}) \cdot \vec{k}\, dA = \int_{\partial D} \vec{v} \cdot d\vec{r}}$ and

$\underline{\int_{\partial D} \vec{v} \cdot (y'(t) i - x'(t) j) \cdot \frac{1}{|r(t)|} ds = \iint_D \text{div}(\vec{v})\, dA}$

$$\text{curl}(\vec{v}) = \det \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle -Q_z, +P_z, Q_x - P_y \rangle$$

$$\text{curl}(\vec{v}) \cdot \underset{(0,0,1)}{k} = Q_x - P_y = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

So first equality is $\iint_D \text{curl}(\vec{v}) \cdot \vec{k}\, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \overset{\text{Green's theorem.}}{=} \int_{\partial D} \vec{v}\, d\vec{r}$

So first equality is $\iint_D \operatorname{curl}(\vec{v}) \cdot \vec{k}\, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \overset{\dagger}{=} \int_{\partial D} \vec{v}\, d\vec{r}$

$$\operatorname{div}(\vec{v}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, 0 \rangle$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

v.f $\vec{w} = \langle -Q, P, 0 \rangle$

$$\iint_D \operatorname{div}(\vec{v})\, dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

$$= \int_{\partial D} \vec{w} \cdot d\vec{r}$$

Green's theorem

$$= \int_{t=a}^{b} \left(-Q x'(t) + P y'(t)\right) dt$$

$$= \int_{t=a}^{b} \langle P, Q \rangle \cdot \langle y', x' \rangle\, dt$$

$$= \int_{\partial D} \vec{v} \cdot \left(y'(t) i - x'(t) j\right) \frac{1}{|r'(t)|}\, ds.$$

Stoke's theorem

divergence

Point: Green's theorem can be recast using either ① curl or ② divergence. Theorem

These two ways of recasting Green's theorem lead to two separate generalizations of Green's theorem.

final

§16.6 Parametric Surface

Definition: A parametric surface is a function $\vec{s}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ for some domain $D \subseteq \mathbb{R}^2$

Idea: This is a " space curve of dimension 2"

Ex. A sphere of radius $r > 0$ can be parametrized as :

$$S(\theta, \varphi) = \langle r \sin\varphi \cos\theta,\ r \sin\varphi \sin\theta,\ r\cos\varphi \rangle$$

comes from spherical coordinates

on $D = [0, 2\pi] \times [0, \pi]$

Ex. The torus has parameterization

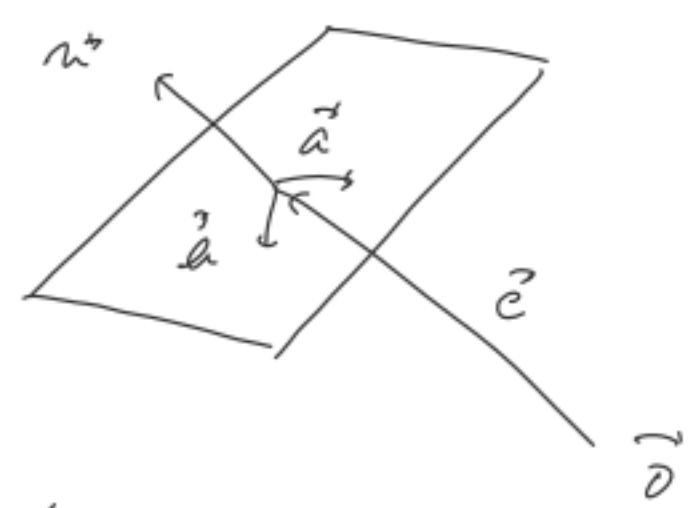
$$\vec{S}(u,v) = \langle 2 + \sin v \cos u,\ (2 + \sin v) \sin u,\ \cos v \rangle$$

on $D = [0, 2\pi] \times [0, \pi]$

Ex. every plane in $\mathbb{R}^3$ can parameterized via

$$\vec{S}(u,v) = u\vec{a} + v\vec{b} + \vec{c} \quad \text{for suitable } \vec{a}, \vec{b}, \vec{c}$$

for $D = \mathbb{R}^2$



Idea: π is just determined by point (u,v) in $\mathbb{R}^2$ via $\vec{a}, \vec{b}, \vec{c}$ & the eq above.

Ex. Compute a parameterization for the paraboloid $z = x^2 + 2y^2$

Note: there are many ways to parameterize this surface.

sol ① $\vec{S}(x,y) = \langle x, y, x^2 + 2y^2 \rangle$ $D = \mathbb{R}^2$

sol ② $\vec{S}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2(1 + \sin^2\theta) \rangle$

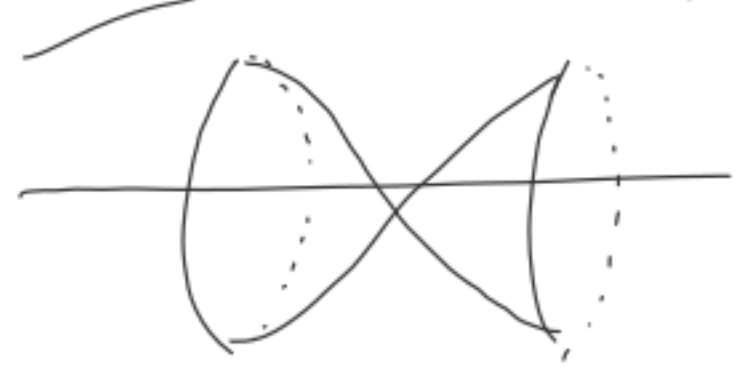
$D = [0, \infty] \times [0, 2\pi]$

sol ③ $\vec{S}(r,\theta) = \langle \sqrt{2} r\cos\theta, r\sin\theta, 2r^2 \rangle$

Ex. Let $f(x)$ be a single-variable function. The surface determined by revolving f about the x-axis is parameterized by

$$\vec{S}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$$

on $D = \mathrm{dom}(f) \times [0, 2\pi]$



↳ sub-ex: Let $f(x) = x^3$

this surface has parameterization

$$S(x,\theta) = \langle x, x^3\cos\theta, x^3\sin\theta \rangle$$